Convergence Characteristics of Frequency-domain LMS Adaptive Filters

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Abstract: The frequency-domain implementation of the adaptive LMS algorithm offers several advantages over the
time-domain LMS algorithm. The orthogonality of the individual bins in the FFT of a signal permits the implementation of
independent complex LMS filters for each FFT bin with separate convergence parameters for each bin. The result is
independent rates of convergence for each FFT bin. The time-domain LMS algorithm converges only as fast as the dominant
eigenvalue of the input data. An additional parameter which affects frequency-domain LMS convergence is the overlap
between successive FFT buffers. The maximum advisable overlap between FFT buffers is 50%; greater overlap between FFT
buffers can lead to oscillations and even instability in the adaptive algorithm.

THE FREQUENCY-DOMAIN LMS ALGORITHM

The standard time-domain least mean squares (LMS) adaptive filter update for a finite impulse response (FIR) filter
with filter coefficients $h_k, k=0,1,2,\ldots,M$ at time $n$ is

$$h_{k,n} = h_{k,n-1} - 2\mu x_{n-k} e_n$$

(1)

where $x_{n-k}$ are past inputs to the filter, $e_n$ is the current error between the measured and predicted filter outputs and $\mu$
is the LMS adaptive filter step size. The parameter $\mu$ can be expressed as $\mu = \mu_{rel}/(\sigma^2_x)$ where $\sigma^2_x$ is the variance of
the input data $x_n$. The step size component $\mu$ creates an effective "data memory window" with an exponential forgetting
property of approximate length $1/\mu_{rel}$. The shortest possible memory window length for stable time-domain LMS
adaptation is the filter length $M+1$ and corresponds to a maximum value of $\mu_{rel} = 1/(M+1)$.

The entire vector of $M+1$ FIR filter coefficients in (1) can be updated at once by expanding the LMS filter update
equation to include blocks of $M+1$ input and error data samples. Letting $H_n$ denote the vector of $M+1$ FIR filter
coefficients at time $n$, the block time-domain LMS filter update equation can be written as

$$H_n = H_{n-N_e} - 2\mu R^e_n$$

(2)

where $N_e$ is the number of samples between block updates and $R^e_n$ is a vector of cross correlations between $n$ samples
of the reference and error signals at times in the block update interval. By taking the Fourier Transform of the block
LMS update in (2), the filter update can be expressed in the frequency domain as

$$H_n(\omega) = H_{n-N_e}(\omega) - 2\mu X^*_{n}(\omega) E_{n}(\omega)$$

(3)

where $H_n(\omega)$ denotes the Fourier transform of the filter coefficients at time $n$, and $X_{n}(\omega)$ and $E_{n}(\omega)$ denote the Fourier
transforms of the reference and error signals respectively.

Caution must be used in computing the cross spectrum in equation (3) due to the effects of circular correlation when
the input signal has sinusoids not bin-aligned with the Fourier transform. The problem occurs because the finite length
signal buffers in the FFT are assumed to be periodic with a period equal to the length of the signal buffer. This is not
a problem for random noise signals or sinusoidal signals where the frequencies lie exactly on one of the FFT frequency
bins. The implementation of a frequency-domain algorithm that is immune to circular correlation errors requires
doubling the FFT buffer sizes, and zero padding one buffer [1]. This shifts the circular correlation errors to half of the
$2(M+1)$ element FIR coefficient vector produced by inverse transforming the transfer function $H_n(\omega)$ [2].

One advantage of the FDLMS algorithm is that the orthogonality of the individual bins in the FFT permits the
implementation of independent single complex LMS filters for each FFT bin. A separate step size parameter is used
for each FFT bin which is inversely proportional to the signal power in the bin. The parameter $\mu_{rel}$ can be set to unity
since the memory window need not be longer than the integration already inherent in the FFT's. Spectral averaging of
the power for the bin step size parameter improves robustness.
ADAPTIVE FILTER EXAMPLE

Figure 1 shows the time response of the error signals from three adaptive filters: recursive least squares (RLS), LMS, and FDLMS. The input signal consists of three sinusoids with f = 50, 250, and 390 Hz, and amplitudes 10, 30, and 20 respectively plus white noise. The RLS algorithm is included as a benchmark case since the LMS and FDLMS are approximations of it. The RLS algorithm converges fastest, followed by the FDLMS, then the LMS algorithm. Figure 2 shows the convergence of each tone in the test signal for each adaptive filter. The RLS and FDLMS algorithms exhibit uniform rates of convergence while the time-domain algorithm converges only as fast as the dominant eigenvalue of the input data. With the LMS filter the frequency components converge at rates proportional to the relative amplitudes of the components. The uniform convergence rate with the FDLMS algorithm is due to the frequency-dependent step size, μ(ω).

An additional parameter which affects convergence of the FDLMS filter is the overlap between successive FFT buffers. The maximum advisable overlap between FFT buffers is 50%. Greater overlap between FFT buffers can lead to oscillations (and even instability) in the adaptive algorithm. Figure 3 shows the effect of larger FFT buffer overlap on FDLMS convergence - the filter begins to converge, then becomes unstable. The FDLMS example in Figure 1 on the other hand uses nonoverlapping FFT buffers.

The FDLMS algorithm offers improved convergence characteristics compared to the time-domain LMS algorithm. The ability to control the adaptive filter step size at each FFT frequency bin leads to uniform rates of convergence at all frequencies. The overlap between FFT buffers also affects FDLMS convergence and can lead to instability if the overlap is too large.

REFERENCES