Determination of roots of eikonal equation for WKB solution in cochlear models

Anand A. Parthasarathi\textsuperscript{a}, Karl Grosh\textsuperscript{b}, and Alfred L. Nuttall\textsuperscript{c}

\textsuperscript{a}Dept. of Biomedical Eng., University of Michigan, Ann Arbor, MI 48109, \textsuperscript{b}Dept. of Mech. Engin. and Appl. Mechanics, University of Michigan, Ann Arbor, MI 48109, and \textsuperscript{c}Oregon Hearing Research Center, Oregon Health Sciences Univ., Portland, OR.

Abstract: WKB theory based asymptotic methods have been used to obtain solutions to analytical cochlear models (2). The local wavenumber for this solution is obtained from the root of the eikonal equation. For a three-dimensional rectangular model, multiple complex roots exist. However, previous results were based on a single root. In this study, the winding-number integral technique is used to obtain a definitive identification of the multiple roots and to study their influence on the cochlear traveling wave. It was found that for a 3-D model this technique could exactly determine the root locus beyond basilar membrane (BM) resonance location, which is where techniques based on Newton-Raphson (NR) iterations fail. However, a comparison of WKB and FEM results predict that more than one locus may be required to obtain a more accurate solution.

INTRODUCTION

Closed-form solution for three-dimensional cochlear models are typically not possible, and hence most modelers obtain asymptotic solutions. A popular solution technique based on the WKB theory has been discussed extensively by de Boer, among others. However, results predicted by this technique fall short of expectations, especially close to the resonance location of the BM. For a 3-D passive cochlear model (along the ideas of de Boer) one could obtain a WKB approximation for the BM velocity, \( w(z) \), by assuming a dispersive wave and slowly varying BM impedance, \( \sigma(z) \). It turns out to be,

\[
 w(z) = C_e \left( \frac{dQ(k_z)}{dk_z} \right)^{-\frac{1}{2}} e^{-i \int_0^z \sigma(u) du} 
\]  

(1)

In the above equation \( Q(k_z) \) is a kernel that depends on the assumed shape of the cochlea and \( k_z \) is the local wave-number, which is determined as a root of the eikonal equation,

\[
 \zeta(z) = -2i \omega \rho Q(k_z) 
\]  

(2)

This is a transcendental equation, with infinite possible roots. de Boer has proposed a solution based on the root locus originating from close to the origin of the complex \( k_z \) plane. Fig. 1 shows the BM velocity determined using this technique as compared to the results from our finite element analysis.

MULTIPLE ROOTS

The response predicted by the WKB method breaks down close to resonance since the root locus could not be determined beyond resonance using NR iteration. Brazier-Smith and Scott (1) have suggested a simple technique for root determination, whereby the location of poles and zeros could be accurately determined using higher moments of the winding-number integral (WI). Fig. 2 shows the root locus determined using this technique. The WI method determines the locus even beyond resonance, while NR fails at resonance. However, BM velocity predicted using this 'extended locus' still dies out close to resonance. This would mean that additional root loci, and hence traveling waves, will have to be used to obtain a better WKB
Figure 1. BM velocity obtained using WKB approximation and FEM.

Figure 2. Primary root locus determined using NR and WI. Also shown is the higher root.

approximation. In fig. 2 a smaller locus originating at $k_p = -i \pi / H$, where $H$ is height of the cochlear duct, is also shown. In fact, each of the pole located at multiples of $k_p$ gives rise to a locus. In his thesis, Watts (3) has used the first higher root to develop a mode-coupled LG solution which provides a good approximation. An alternative method would be to approximate $Q(k_z)$ using the pole-zero pairs close to the origin and obtain higher order differential equations in BM pressure. Asymptotic solutions to these ODEs could then be determined. However, for a 3-D model the order of the differential equation is very high, and hence a better solution technique has to be developed.

REFERENCES

