New experimental characterization of a resonance: identification of the mode number using the Argand diagram and the GTD approach

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Abstract: A single experiment is carried out on a cylindrical elastic target immersed in water and insonified at normal incidence by an incident plane wideband wave. The normalized backscattered pressure (ratio of the FFT of the sampled backscattered signal to the FFT of the incident waveform) is analyzed in the complex pressure plane, involving on one hand the Resonant Scattering Theory, and on the other hand, the Geometrical Theory of Diffraction. Once the parameters of each resonance (frequency, width, phase lag) are measured, the mode number is identified.

MEASUREMENT OF RESONANCES PARAMETERS OF A CYLINDRICAL TARGET

An air-filled aluminum cylindrical shell ($b/a=0.9$, where $b$ and $a=3.6$ mm are the shell internal and external radii) imbedded in water, is insonified at normal incidence by a short ultrasonic pulse generated by a wideband transducer (frequency range: 500 kHz-4 MHz). The backscattered pressure, received by the same transducer is recorded during 500 µs and sampled by a 10-bits DSO at a sample rate equal to 100 MHz. The complex form function $P_{diff}(f)$ of the shell is calculated by dividing the FFT of the previous signal by the FFT of the incident waveform ($\times=k_1a$ normalized frequency, $k_1$: wave number in the external fluid). The resonances ($A$ and $S_0$ wave resonances, in the present case) of the shell are easily isolated by pointing out the maxima of the plot of $|\delta P_{diff}(f) / \delta f|$, modulus of the numerical derivative of $P_{diff}(f)$ (1); indeed, at the neighbourhood of the frequency $f_{n_{th}}$ of the $n_{th}$ resonance in the $n_{th}$ mode, the Resonant Scattering Theory (2) allows us to write $P_{diff}(f)$ (i.e. the Rayleigh series) as the sum of a background pressure term $P_{back}$ and a resonance term $P_{res}$ described by the Breit-Wigner function

$$P_{diff}(f) = P_{back} + P_{res} = \frac{P_{back} + P_{n} e^{i\phi_n} e^{i f n / 2}}{x - x_n + j \Gamma_n / 2}$$

where $\Gamma_n$ is the resonance width, $P_n$ is the resonance amplitude, and $\phi_n$ the phase lag near the resonance maximum. Here, $P_{back}$ includes the whole non-resonant contribution and all other resonant contributions different and supposed far from the currently isolated one ($n,t$). Although $P_{back}$, $P_n$, and $\phi_n$ strictly depend on the frequency $x$, their spectral variations can be considered as negligible compared to the resonant term variations. Figure 1 illustrates the previous decomposition in the complex pressure plane (or Argand plane): small circles represents sampled values of $P_{diff}(f)$ for equally spaced values of $x$. The data plot exhibits the typical circular shape characterizing the Breit-Wigner function. The resonance maximum is reached at point R (when $|\delta P_{diff}(f) / \delta f|$ is maximum). Points Q and S represent the $P_{diff}(f)$ values corresponding to the frequency abscissae $x_n - \Gamma_n / 2$ and $x_n + \Gamma_n / 2$ respectively and verify $\overrightarrow{RQ} = \overrightarrow{RS} = \pi / 2$. The unknown parameters $P_{back}$, $P_n$, $\phi_n$, and $\Gamma_n$ are fitted using a Least Mean Squares algorithm, involving the pressure values included in the Q to S points data range. As an example, measured values of $\Gamma_n$ related to $A$ wave resonances are given in Table 1. The resonance circle and the vectors plotted in dashed lines on Figure 1 result from the calculation of the $P_{diff}(f)$ (see equation (1)), using the fitted parameters. On Figure 2, the phase lag $\phi_n$ is
reported versus $x_{nr}$ for each resonance isolated in the experimental frequency range; black diamonds represent $S_0$ wave resonances, and squares represent $A$ wave resonances.

An analytical approximation of $\varphi_{nr}$ as a function of $n$ is given by applying the Sommerfeld-Watson transform (SWT) to the exact total pressure field (3). It consists in replacing the normal mode series by a contour integral in the complex $\nu$-plane ($\nu$ is the complex expansion of $n$) under high frequencies hypothesis. Introducing the Imai separation that allows, in the previous integral, to isolate the surface waves contribution from the geometrical waves contribution, and carrying out the appropriate asymptotic expansion of the involved Bessel and Hankel functions, the value of a surface wave phase $\varphi_{nr}$ at the resonance frequency $x_{nr}$ is expressed as

$$\varphi_{nl} = \pi / 4 + 2\nu_{nl} - 2\nu_{nl} (1 - (\nu' / x_{nl})^2)^{1/2} - 2n \cos^{-1}(\nu' / x_{nl}) - (\nu' - 1)\pi$$

where $\nu'$ (real part of $\nu$) coincides with the mode number $n$ (fulfilling then the phase matching condition), and $\sin(\vartheta) = \nu' / x_{nr}$ with $\vartheta$ defined as the excitation angle of the surface wave. The equality is not strict because the background phase has been neglected here. The mode number associated to each $S_0$ resonance is calculated by minimizing the error $|\varphi_{nr} - \varphi_{nl}|$ using trial values of $n$ from 0 to $\text{Int}(2x_{nr})$. The result is plotted with a solid line on Figure 2. One can notice the good agreement between the measured phase and the fit; errors are mainly due to the background phase term, which has not been taken into account. Relation (2) is only valid in the case of internal surface waves ($n < x_{nr}$) and cannot strictly be used to describe external surface waves such as A wave ($n > x_{nr}$). But considering that $\vartheta$ is complex in this last case, the phase can be rewritten here

$$\varphi_{nl} = \text{Re}[\pi / 4 + 2\nu_{nl} - 2\nu_{nl} \cos(\vartheta) - 2n \cos^{-1}(\sin(\vartheta)) - (\nu - 1)\pi]$$

where now $\sin(\vartheta) = (\nu' + j\nu') / x_{nr}$. So, in order to determine the mode number associated to A-wave resonances, the imaginary part $\nu''$ of the mode number has to be estimated first. Considering the Breit-Wigner relation

$$\nu''(x_{nl}) \approx (\Gamma_{nl} / 2)(\partial \nu' / \partial x)_{x_{nl}}$$

and identifying $(\partial \nu' / \partial x)_{x_{nl}}$ as the inverse of the mean frequential step between two successive A resonances, $\nu''$ are evaluated (Table 1). Then the same modus operandi than in the case of $S_0$ resonances is carried out, but involving now relation (3). As it can be seen in Figure 2, the agreement is very good.

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**TABLE 1.** Measured frequencies, widths, and estimation of $\nu''$ for A wave resonances.

**FIGURE 2.** Resonance phase lag versus resonance frequencies.

**REFERENCES**