Simple model for temperature gradient formation in a short stack

Ralph T. Muehleisen¹ and Anthony A. Atchley²

¹ Department of Civil, Environmental, and Architectural Engineering, University of Colorado, Boulder CO, 80309
² Graduate Program in Acoustics, Penn State University, University Park, PA, 16802

Abstract: A model for the time evolution of the temperature in a stack plate is developed assuming a thermoacoustic temperature flux as developed by Arnott [1] entering and leaving the plate only at the very edges. The model compares fairly well to measurements.

ANALYSIS

Many analytic and numerical tools have been developed for the analysis of steady state thermoacoustics phenomenon. Recently some numerical and experimental work has appeared [2,3], but little analytic work has been done. In this paper a simple model for the temperature evolution in a single plate of a stack is presented. Since the period of the acoustic oscillations is much shorter than the time constants related to plate heating, the thermoacoustic phenomenon is essentially in steady state with respect to the heating in the plate. Thus, the steady state heat flux as derived by Arnott [1] can be assumed to drive the plate heating. While a recent paper by Mozurkewich details the spatial dependence of the heat flux [4], the model is too complex for this analysis. In this model, the thermoacoustic heat flux is assumed to enter and leave the plate at the very edge.

Consider a stack of plates of length L, thickness d, spaced h apart, with a density ρ, thermal conductivity κ, and specific heat C_p placed at a position x from the end of a tube length L_t. In the tube is a standing wave acoustic field with acoustic pressure P_a, ambient pressure P_0, and frequency ω in a gas with thermal conductivity κ_g, speed of sound c, density ρ_g, and specific heat C_p_g. The thermoacoustic heat flux across the plate will be [1]

\[ \dot{Q} = \left\{ \begin{array}{ll} AP_a^2 + (BP_a^2 + C) \frac{dT}{dx} & 0 < x < L \\ 0 & \text{otherwise} \end{array} \right. \]

where

\[ A = \frac{2h \sin 2k(L_t - x)}{4\rho_p c(1 + \sigma)} \left\{ F^*(\lambda_t) \right\} \left\{ F(\lambda) \right\}, \]

\[ B = -A \frac{\text{Im}\left\{ F(\lambda) + \sigma F^*(\lambda) \right\}}{\text{Im}\left\{ F(\lambda)F^*(\lambda) \right\}(1 - \sigma)} \frac{C_p_g \tan k(L_t - x)}{\rho_g c}, \]

\[ C = -2 + 6k + \frac{\kappa_g}{\rho_g C_p_g \omega}, \]

\[ \delta_k = \sqrt{2k_g/(\rho_g C_p_g \omega)} \]

\[ \frac{d^2T}{dx^2} - \frac{\rho \kappa_p d}{d + (C + BP_a^2)}/ \frac{dT}{dt} \]

The plate heating is governed by the heat equation

\[ \frac{\partial \dot{Q}}{\partial x} + \kappa d \frac{\partial^2 T}{\partial x^2} = \rho C_p d \frac{\partial T}{\partial t}. \]

The first term of Eq. 5 can be expanded as

\[ \frac{\partial \dot{Q}}{\partial x} = (AP_a^2 + (BP_a^2 + C) \frac{\partial T}{\partial x} \left[ \delta(x) - \delta(x - L) \right] + (BP_a^2 + C) \frac{\partial^2 T}{\partial x^2}. \]

Since the delta functions act at the boundaries of the plate, the partial differential equation can be rewritten as a boundary value problem of the form

\[ \frac{\partial^2 T}{\partial x^2} = \frac{\rho C_p}{\kappa + (C + BP_a^2)}/ \frac{dT}{dt} \]

with

\[ \frac{\partial T}{\partial x} |_{x=0,L} = \frac{-AP_a^2}{\kappa d + 2(C + BP_a^2)}. \]
A solution to this equation is

\[ T = T_0 + \frac{A P_a^2}{2 k_1 d} (L - x) + \frac{4 A P_a^2}{k_1 d L} \sum_{m=1,3,5,\ldots}^{\infty} \frac{\cos\left(\frac{m\pi x}{L}\right)}{\left(\frac{m\pi}{L}\right)^2} e^{-K_{eq}\left(\frac{m\pi}{L}\right)^2 t} \]  

(9)

where \( K_{eq} = \frac{(\rho C_p)}{(K_1 + (C + B P_0^2)/d)} \) and \( k_1 = k_0 + 2(C + B P_0^2)/d \).

**COMPARISON TO MEASUREMENT**

Figure 1 shows a comparison between the model and measurements of the temperature at the edge of 0.28 mm thick stainless steel plates spaced 2.5 mm apart in argon at 297 K and \( P_o = 2.1 \times 10^4 \text{ Pa} \) with \( P_a/P_o = 4.1\% \) and \( P_a/P_o = 8.6\% \). To compare the graphs the viscous heating (as measured by a thermocouple at the center of the plate) has been removed from the measurement. Measurement and theory match well at 4.1\% and fairly at 8.6\%. At 4.1\% the theory predicts both the amplitude and time constant well. At 8.6\% the theory over predicts the time constant and under predicts the amplitude but the difference is within the error usually found between thermoacoustic theory and measurements.

**CONCLUSIONS**

A simple model for the time evolution of the temperature of a stack plate has been developed. The model assumes a steady state thermoacoustic heat flux that enters and exits the plates only at the very edges. The model is shown to match measurements fairly well. Application of Mozurkewich's new model for spatial dependence might help reduce the disagreement at higher pressure ratios.

**ACKNOWLEDGEMENTS**

This work was supported by the Office of Naval Research and the American Association for Engineering Education.

**REFERENCES**