Complex resonances of elliptic cylinders: numerical evaluation and full classification

Stéphane Ancey*, Antoine Folacci and Paul Gabrielli

URA 2053 CNRS, Equipe Ondes et Acoustique, Université de Corse, 20250 Corte, France.
*e-mail: anc@univ-corse.fr

Abstract:
Complex resonances of elliptic cylinders of arbitrary eccentricities are exactly determined by solving the Helmholtz equation with the Sommerfeld's radiation condition at infinity and various boundary conditions on the scatterer. Resonances are then fully classified according to the four irreducible representations $A_1$, $A_2$, $B_1$ and $B_2$ of the symmetry group $C_{2v}$ of the scatterer. This spectrum of resonances is partially recovered by using the ordinary Geometrical Theory of Diffraction. Unfortunately, it fails to provide a physical interpretation of the splitting up of resonances, obtained from group theory considerations, because of its purely local nature.

GENERAL THEORY

The geometry of an elliptic cylinder is described by elliptic coordinates $(u, v)$ related to rectangular cartesian coordinates $(x, y)$ by the transformation

$$
x = c \cosh u \cos v \quad (1)
$$

$$
y = c \sinh u \sin v \quad (2)
$$

where $0 \leq u < +\infty$, and $0 \leq v < 2\pi$ (see Fig. 1). $u = u_0$ defines the surface of the scattering body. Its eccentricity is equal to $e = 1/ \cosh u_0$. The limit cases $u_0 = 0$ and $u_0 \rightarrow +\infty$ correspond respectively to the strip and the circular cylinder. In the elliptic cylinder case, the invariance of the circular cylinder under the continuous group $O(2)$ is broken, but this scatterer is however invariant under a finite group. Indeed, the geometry of the elliptic cylinder, as shown in Fig. 1, is invariant under four symmetry transformations: i) $E$, the identity transformation, ii) $C_2$, the rotation through $\pi$ about the origin $O$ of the elliptic coordinates, iii) $\sigma_x$, the mirror reflection in the $Ox$ axis, and iv) $\sigma_y$, the mirror reflection in the $Oy$ axis. These four transformations form a finite group of order 4, labelled $C_{2v}$ in the mathematical literature, (see for example the book of Hamermesh (1)), which can be called the symmetry group of the scatterer. Four one-dimensional irreducible representations labelled $A_1$, $A_2$, $B_1$, $B_2$ are associated with this symmetry group $C_{2v}$. In the study of acoustic scattering by an elliptic cylinder, it seems then natural to classify the resonances according to these four irreducible representations.

In elliptic coordinates, the resonant modes are searched by separation of variables on the form $U(u)V(v)$ where $U$ and $V$ satisfy the Mathieu equations

$$
\frac{d^2V}{du^2} + (h - 2\theta \cos 2v) V = 0 \quad (3)
$$

$$
\frac{d^2U}{du^2} - (h - 2\theta \cosh 2u) U = 0 \quad (4)
$$

as well as the Sommerfeld's radiation condition at infinity and various boundary conditions on the scatterer. Here $h$ is a separation constant and $\theta$ is linked to the wave number $k$ by $\theta = (kc/2)^2$. Symmetry considerations permit us to
select four sets of characteristic values of the separation constant \( \lambda \) denoted in the literature (2) by

\[
(a_{2r} (\theta))_{r \in \mathbb{N}}, (a_{2r+1} (\theta))_{r \in \mathbb{N}}, (b_{2r+2} (\theta))_{r \in \mathbb{N}}, (b_{2r+1} (\theta))_{r \in \mathbb{N}},
\]

and respectively associated with the irreducible representations \( A_1, B_1, A_2, B_2 \) of \( C_{2v} \). The previous problem is then numerically solved.

**NUMERICAL RESULTS**

We present in Fig. 2 the locations of resonances in the complex \( ka \)-plane for soft and hard elliptic cylinders. Here the reduced wave number is defined by \( ka = (kc/2) \exp u_0 \). For \( u_0 \rightarrow +\infty \) (and \( kc \rightarrow 0 \)), it corresponds to the usual reduced wave number used in the context of scattering by a circular cylinder of radius \( a \). Scattering resonances are plotted for different values of \( u_0 (u_0 \rightarrow +\infty, u_0 = 1, u_0 = 0.5, u_0 = 0.2, u_0 = 0.1, u_0 = 0.05) \). The splitting up of resonances linked to the breaking of the \( O(2) \)-symmetry is thus displayed. Some numerical examples are also presented in Table 1 and Table 2.

**FIGURE 2.** Location of the scattering resonances in the complex \( ka \)-plane for soft (a) and hard (b) elliptic cylinders. Resonances of the circular cylinder are denoted by (o) and are plotted for \( l = 1 \) and from \( n = 4 \) to 15. Resonances of the representations \( A_1, A_2, B_1, B_2 \) are respectively denoted by (*), (.), (x), (+).

**Table 1.** Resonances for soft elliptic cylinders.

<table>
<thead>
<tr>
<th>((n, l))</th>
<th>((6, 1))</th>
<th>((7, 1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(u_0 \rightarrow +\infty)</td>
<td>(4.0310-2.4234i)</td>
<td>(4.9560-2.6031i)</td>
</tr>
<tr>
<td>(u_0 = 1)</td>
<td>(4.0316-2.4099i)</td>
<td>(4.9539-2.5866i)</td>
</tr>
<tr>
<td>(u_0 = 0.5)</td>
<td>(4.0357-2.3259i)</td>
<td>(4.0329-2.3256i)</td>
</tr>
<tr>
<td>(u_0 = 0.2)</td>
<td>(4.0606-2.0992i)</td>
<td>(3.9930-2.1386i)</td>
</tr>
<tr>
<td>(u_0 = 0)</td>
<td>(4.0044-1.6373i)</td>
<td>(\times)</td>
</tr>
</tbody>
</table>

**Table 2.** Resonances for hard elliptic cylinders.

<table>
<thead>
<tr>
<th>((n, l))</th>
<th>((6, 1))</th>
<th>((7, 1))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(u_0 \rightarrow +\infty)</td>
<td>(5.2366-1.2383i)</td>
<td>(6.2002-1.3070i)</td>
</tr>
<tr>
<td>(u_0 = 1)</td>
<td>(5.2209-1.2278i)</td>
<td>(5.2210-1.2277i)</td>
</tr>
<tr>
<td>(u_0 = 0.5)</td>
<td>(5.1179-1.1654i)</td>
<td>(5.1273-1.1603i)</td>
</tr>
<tr>
<td>(u_0 = 0.2)</td>
<td>(4.8109-1.0764i)</td>
<td>(4.9144-0.9411i)</td>
</tr>
<tr>
<td>(u_0 = 0)</td>
<td>(\times)</td>
<td>(4.5049-0.5918i)</td>
</tr>
</tbody>
</table>

Geometrical Theory of Diffraction permits us to partially recover the previous spectra. For soft (respectively hard) elliptic cylinders we obtain the resonances associated with the irreducible representation \( A_1 \) and \( B_1 \) (respectively \( A_2 \) and \( B_2 \)). The local nature of the attenuation coefficient used in GTD fails to explain the splitting up of resonances.

The numerical determination of these scattering resonances is useful in order to synthesize the time signature of elliptic cylinders. The splitting up of resonances could be directly observed on temporal echoes.

**REFERENCES**