Finite-Element Modelling of the Vibro-Acoustical Behaviour of Poro-Elastic Materials

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Abstract: Porous materials are widely used nowadays as part of insulation systems in the automotive industry or as effective sound absorbers in rooms. Their complex dynamic behaviour is due to the various interaction phenomena within the elastic-porous material. Such materials have been effectively modelled, thanks to the original work of Biot (1), extended by Alland (2). In the work reported here, the classical Biot theory for elastic porous materials is used in the development of a three-dimensional finite element. While the finite-element development remains classical, a new numerical implementation that allows shorter computation times is proposed.

SOUND PROPAGATION IN POROUS MATERIALS

Assuming a harmonic time dependence of the form $e^{i\omega t}$, the two coupled, vectorial differential equations that govern wave propagation in poro-elastic media can be written as:

\[ N\nabla^2 \mathbf{u}_s + \nabla[(A+N)\varepsilon_s + Q\varepsilon_f] = -\alpha^2(\rho_1 u_s + \rho_2 u_f) + j\omega(b \mathbf{u}_s - \mathbf{u}_f) \]  

\[ \nabla[Q \varepsilon_f + R \varepsilon_s] = -\alpha^2(\rho_1 u_s + \rho_2 u_f) - j\omega(b \mathbf{u}_s - \mathbf{u}_f) \]

where $\varepsilon_s = \nabla \mathbf{u}_s$ is the solid-phase volumetric strain, $\varepsilon_f = \nabla \mathbf{u}_f$ is the fluid-phase volumetric strain, $\mathbf{u}_s$ is the solid-phase displacement field, $\mathbf{u}_f$ is the fluid-phase displacement field, $A$ is the first Lamé constant and $N$ is the elastic shear modulus. The quantities $\rho_1$, $\rho_2$ and $\rho_{22}$ in Biot theory (1) are related to the density of the frame $\rho_1$, the density of air $\rho$, the mass coupling term $\rho_2$ and the porosity $\phi$ of the material.

The three coefficients $P$, $Q$ and $R$ can be determined by two gedanken experiments (2). Hence, for materials with homogeneous elastic frame, these coefficients are given by (2):

\[ P = [(1-\phi)(1-\phi-K_s/K_f)K_s + \phi(K_s/K_f)K_f][1-\phi-K_s/K_f]^{-1} + 4N/3 \]  

\[ Q = [(1-\phi-K_s/K_f)\phi K_s][1-\phi-K_s/K_f + \phi(K_s/K_f)]^{-1} \]  

\[ R = [\phi^2 K_f][1-\phi(K_s/K_f) + \phi(K_s/K_f)]^{-1} \]

These equations are valid for all isotropic poro-elastic materials having a homogeneous frame. $K_s$, $K_f$ and $K_t$ are, respectively, the bulk moduli of the frame material, the frame at constant pressure and of air. For numerous other porous materials, such as glass wools or reticulated foams, the porosity $\phi$ is close to 1 and $K_s/K_f$ is much smaller than $(1-\phi)$, so these equations can be simplified (2).

FINITE-ELEMENT MODELING

The classical finite-element method has been used to develop 3-D and 2-D poro-elastic finite elements based on Biot theory (1). The finite-element development itself is not complicated for the expert in the field of numerical methods. However, the computation times required are usually constraining due to the fact that the poro-elastic
finite elements have up to 6 degrees of freedom per node (4 degrees of freedom for 2-D elements) if one uses displacement fields as unknowns. It is possible to reduce the number of degrees of freedom by using the pressure \( P \) instead of displacements for the fluid-phase of the porous material, although an alternative technique for fast computation times still seems to be necessary.

The finite-element algebraic system obtained has the following form:

\[
\begin{bmatrix}
\tilde{Z}_p(\omega) & C^T \\
C & Z_\omega
\end{bmatrix}
\begin{bmatrix}
u_p \\
x_\omega
\end{bmatrix}
= \begin{bmatrix}
F_p \\
F_\omega
\end{bmatrix}
\]  

where \( \tilde{Z}_p(\omega) \) is the porous global dynamic matrix which is complex and frequency dependent, \( \langle \mathbf{u}_p \rangle = \langle \mathbf{u}_\omega, \mathbf{u}_p \rangle \), \( Z_\omega \) is the global dynamic matrix of a given 'other' medium coupled to the porous material, \( x_\omega \) is its unknowns vector, and \( F_p \) and \( F_\omega \) are excitations. The main reason for which computation times are long is that the \( \tilde{Z}_p(\omega) \) matrix is frequency dependent. This dependence is due to the complex and frequency-dependent bulk modulus of air \( K_f(\omega) \).

**PROPOSED SOLUTION**

The real and imaginary parts of the bulk modulus of air \( K_f \) generally behave simply as shown in Figure 1 for the example of a fibrous material. This allows the frequency range to be divided into a given number of intervals in which \( K_f \) is constant. Only two intervals are considered here for the sake of simplicity - first interval \([0,2500]\) Hz and second interval \([2500,5000]\) Hz.

For each interval \( \mathcal{I} \), the porous global matrix becomes frequency-independent; thus it is possible to construct a porous modal basis \( \{\Psi_i\} \). The expression for the porous material's global matrix \( \tilde{Z}_p \) over the modal basis \( \{\Psi_i\} \) leads to a small algebraic system, which is frequency-independent over the interval \( \mathcal{I} \). One must be careful around the junctions because of the discontinuity (indicated by an arrow). Instead of assembling the global matrix \( \tilde{Z}_p(\omega) \) for each calculation-frequency, one needs only to assemble a small, frequency-independent global matrix that can be read from a file. The price to pay is a small reduction of accuracy; this is not very significant as the whole theory is based on a macroscopic description that is itself not very accurate.

![Figure 1. Frequency dependence of the bulk modulus of air: \( K_f \) (\( P_0 \) is the ambient mean pressure).](image)

**REFERENCES**