Interaction of Counterpropagating Acoustic Waves in Materials with Nonlinear Dissipation and in Hysteretic Media

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Abstract: The interaction of sound waves travelling in the opposite directions in media with quadratic nonlinearity is analyzed theoretically. It is well-known that quadratic elastic nonlinearity does not provide an effective interaction of counterpropagating acoustic waves. In other words, the mutual influence of counterpropagating waves does not accumulate with the propagation distance. One of the research results presented here is rather expected. It consists of the prediction that, in media with quadratic nonlinear dissipation, a wave travelling in one direction induces additional accumulating attenuation of a wave travelling in the opposite direction. More interesting result is the prediction that, in the media with hysteretic quadratic nonlinearity, a strong sound wave travelling in one direction induces accumulating amplification of a weak sound wave travelling in the opposite direction. The threshold for the stimulated backscattering of acoustic waves in the hysteretic media is evaluated.

INTRODUCTION

There is currently an additional interest to the problem of the interaction of sound waves travelling in opposite directions in nonlinear media due to progress in thermoacoustics (where acoustic resonator is a necessary part of both a thermoacoustic refrigerator and a thermoacoustic prime-mover) (1) and due to the achievements in nonlinear diagnostics of solids via the evaluation of standing waves in rods (2). For both of these quite separated areas of fundamental and applied research, the physics of the counterpropagating waves possible interaction is of crucial importance.

DISSIPATIVE NONLINEARITY

A well-known example of the quadratic dissipative nonlinearity is provided by nonlinear acoustic phenomena in the rigid framed gas-filled porous materials (3). For many air-saturated materials of considered structure, there is a convincing evidence (3) that Forchheimer-type nonlinearity (which is related to the dependence of flow resistivity on fluid velocity) is the dominant nonlinearity for the acoustic wave propagating through the fluid. There are also indications that to account for the linear dependence of flow resistivity on flow velocity is of greatest practical interest (3). The corresponding generalization of the linear wave equation is

\[
\frac{\partial^2}{\partial t^2} \tilde{\rho}[v] - \frac{\partial^2}{\partial x^2} \tilde{K}[v] = -\tilde{R}_w \frac{\partial}{\partial t} \langle |v|^2 \rangle,
\]

where \( \rho \) denotes gas particle velocity, \( \tilde{\rho} \) and \( \tilde{K} \) are, in the general case, the linear integro-differential operators over time \( t \), \( \tilde{R}_w \) is a constant positive coefficient which can be determined experimentally by static flow resistivity measurements (3). The dispersion relation between the wavenumber \( k \) and the frequency \( \omega \) of small-amplitude (i.e. linear) acoustic wave is obtained by substituting a pure tone (\( \propto \exp(-i\omega t + i k x) \)) in the linearized version of Eq.(1): \( k^2 = \omega^2 \rho(\omega)/K(\omega) = \left\{ \pm [k'(\omega) + i k''(\omega)] \right\}^2 \). Here and later on, \( f' \) and \( f'' \) denote the real and imaginary parts of the arbitrary complex function \( f \). We are searching a solution of Eq. (1) in the form of a sum of counterpropagating harmonic acoustic waves with slowly varying amplitudes \( A_+ \) and phases \( \phi_+ \): \( v = v_+ + v_- = \mu A_+(\zeta) \sin[\omega t - k' x + \phi_+(\zeta)] + \mu A_-(\zeta) \sin[\omega t + k' x + \phi_-(\zeta)] \), \( \zeta \equiv \mu x \). Here \( \mu \ll 1 \) is a small scaling parameter. We restrict here the analysis to the limiting case \( A_-/A_+ \propto O(\mu) \ll 1 \). Finally, by segregating the uni-directional waves with the help of mathematical procedure outlined in (4), we derive the following equations for the slowly varying amplitudes.
Equations (2) describe nonlinear attenuation of a strong pump wave propagating in the positive direction and the nonlinear attenuation of a weak signal wave propagating in the negative direction, induced by a strong pump wave propagating in the positive direction. Thus the theory presented here predicts that, in media with quadratic nonlinear dissipation, the interaction of counterpropagating waves leads to effects accumulating with propagation distance.

HYSTERETIC NONLINEARITY

The simplest wave equation with quadratic nonlinearity, which takes into account the hysteresis in the stress/strain relationship of micro-inhomogeneous materials (such as rocks, microcrystalline metals, ceramics, etc.), can be written in the form (5)

\[
\frac{\partial^2 u}{\partial t^2} - c_0^2 \frac{\partial^2 u}{\partial x^2} = -2c_0^2 \varepsilon \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} - c_0^2 h \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \frac{\partial u}{\partial x} \text{sgn} \left( \frac{\partial^2 u}{\partial x^2} \right) \right] \frac{\partial^2 u}{\partial x^2}. \tag{3}
\]

Here \( h \) is the parameter of the quadratic hysteretic nonlinearity. In the derivation of the second (related to hysteresis) term in the r.h.s. of Eq.(3), it is assumed that the wave process is periodic with an amplitude \((\partial u / \partial x)^A\). The hysteretic stress/strain relationship, leading to this particular form of the hysteretic nonlinearity in Eq.(3), was found experimentally (6). It corresponds to the so-called bow tie dependence of the hysteretic contribution to elastic modulus on strain (7). Separation of the countepropagating waves in Eq.(3) provides

\[
\frac{\partial A_+}{\partial \zeta} \equiv -\frac{1}{2} h \frac{\omega}{c_0^2} A_+, \quad \frac{\partial A_+}{\partial \zeta} \equiv -\frac{2}{3\pi} h \frac{\omega}{c_0^2} A_+^2, \quad \frac{\partial A_-}{\partial \zeta} \equiv -\frac{1}{\pi} h \frac{\omega}{c_0^2} A_+ A_- \equiv k''(A_+)A_. \tag{4}
\]

Equations (4) prescribe that a strong pump wave propagating in the positive direction equally changes its own phase velocity and the phase velocity of a weak signal wave travelling in the opposite direction. In the absence of relaxation processes the hysteretic nonlinear parameter \( h \) is considered to be positive (6, 7). Then, it follows from equations (4) that the velocity of sound propagation diminishes, pump wave exhibits nonlinear attenuation, while a weak signal wave propagating in the negative direction exhibits amplification (induced by a strong pump wave travelling in the opposite direction). If we take into account, like in Eqs.(2), the linear attenuation of the acoustic waves in the material, then the condition \( k'' + k''(A_+) < 0 \) will describe a threshold for the stimulated backscattering of sound. The threshold acoustic Mach number of the strong pump wave is described by \( M_{\text{th}}(A_0) = (\pi / h)(k'' / k'). \) We estimated, for example, that in microcrystalline copper and dry sandstone this threshold value should not exceed \( M_{\text{th}} \) \((10^{-5} - 10^{-6})\).

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References