Description of nonlinear waves in gas-filled tubes by the Burgers and the KZK equations with a fractional derivate

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Abstract: This paper is concerned with an investigation of the nonlinear behavior of sinusoidal plane progressive acoustic waves of finite amplitude in a gas-filled tube. Wave motion is described by the standard and modified Burgers equation or the Khokhlov-Zabolotskaya-Kuznetzov equation which are a very good approximation of the exact equations of fluid motion when effects of nonlinearity and dissipation are relatively small but definitely not negligible. Some approximate solutions of the modified Burgers equation are presented.

In many physical problems involving sound transmission in ducts, the sound-pressure levels involved are so high that the problem of propagation and attenuation cannot be treated using the usual linear acoustic analyses. At these high sound-pressure levels, the nonlinear effects play an important role in attenuation and distortion of the sound waves. These nonlinear effects are of three types: the nonlinearity of the acoustic properties of the lining material, the nonlinearity of the gas itself and the nonlinearity due to the convection. In this paper, we do not consider the nonlinearity of the lining material; that is, we consider waves propagating in hard-walled ducts.

The effect of the nonlinear distortion and attenuation of nonlinear sound plane waves, which propagate in tubes, can be described by the well-known standard Burgers equation ([1]) or the Khokhlov-Zabolotskaya-Kuznetzov (KZK) equation ([3]). Both the equations can be modified if one takes into account the effect of the boundary layer which influences the attenuation and dispersion of sound waves, see [2], [3]. The modified Burgers equation ([2], [3]) can be written in this form:

\[
\frac{\partial v}{\partial x} - \frac{\beta}{c_0^3} \frac{\partial v}{\partial \tau} + \sqrt{2K} \frac{\partial^{1/4} v}{\partial \tau^{1/4}} = \frac{b}{2\rho_0 c_0^3} \frac{\partial^2 v}{\partial \tau^2},
\]

where \( v \) is the particle velocity, \( \tau = t - x/c_0 \) is the retarded time, \( t \) is time, \( x \) is distance from a source, \( \rho_0 \) is density, \( b = 4/3\eta' + \eta'' + \nu(1/c_p - 1/c_v) \) is dissipative coefficient, \( \eta', \eta'' \) are the shear and bulk viscosity coefficients, \( \kappa \) is the heat-conductivity coefficient, \( c_p \) and \( c_v \) are the specific heats at constant pressure and volume, \( \beta = (\gamma + 1)/2 \) is the coefficient of nonlinearity for gases, \( \gamma = c_p/c_v \), \( K = \sqrt{2\nu(1+\gamma-1)}/2c_0R_0 \), \( \nu = \eta'/\rho_0 \), \( Pr \) is the Prandtl's number, \( R_0 \) is inner diameter of the tube. The suffix "0" is used to indicate equilibrium values.

The fractional derivate represents following Blackstock's integrodifferential operator for the acoustic boundary layer (see [3]):

\[
\frac{\partial^{1/4} v}{\partial \tau^{1/4}} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dv(r',x,r=R_0)}{\partial r'} \frac{dr'}{\sqrt{\tau - r'^2}},
\]

where \( r \) is the radial spatial variable.

Inserting the values \( V = v/v_m \) (\( v_m \) is the peak particle velocity), \( \sigma = \beta v_m \omega x/c_0^2 \), \( \gamma = \omega \tau \), \( G_0 = 2\beta v_m \rho_0/c_0^2 \), \( D_0 = K c_0^2 \sqrt{2\gamma} / \omega v_m \beta \) we obtain the convenient nondimensional form of the Eq. (1):

\[
\frac{\partial V}{\partial \sigma} - \frac{1}{2} \frac{\partial V^2}{\partial y} + \sqrt{2D_0} \frac{\partial^{1/4} V}{\partial y^{1/4}} = \frac{1}{G_0} \frac{\partial^2 V}{\partial y^2}.
\]

These model equations can be solved numerically, for instance by means of the Fourier numerical method supplemented by the method of Runge-Kutta of the fourth order or the Adams predictor-corrector method of the second order with an artificial decay because of truncated Fourier series which cause the nonstability of numerical solution. The numerical analysis shows that it is necessary to take into account also the members.
which represent the boundary layer effect in model equations. Therefore the standard Burgers equation (\([1]\)) can be used only in the case of negligible boundary layer effects. The boundary layer significantly influences the form of sound waves, especially due to dispersion. The dispersion makes the asymmetry of the primary symmetrical waves; the peak is rounded while the trough remains sharp. The effects of dispersion grow stronger with distance: the harmonics get more and more out of step with each other.

For a brief analysis it is suitable to use, in the case of sinusoidal waves, these approximate solutions of the Eq. (1):

\[
V = \sum_{n=1}^{\infty} \frac{2J_n[\pi n]}{n} \sin(ny - n\pi D_0 \sigma) \quad \text{for} \quad 0 < \sigma < 1, \tag{4}
\]

where \(A_n(\sigma) = \exp[-\sigma(1/G_0 + n^{-3/2}D_0)]\), \(n\) is the number of given harmonics. The condition \(1/G_0 \sim D_0 \sim v_m/c_0\) must be fulfilled. In case that \(D_0 \sim 10v_m/c_0 > 1/G_0\) then this approximate solution can be used:

\[
V = \sum_{n=1}^{\infty} \frac{2J_n[\pi n B_n(\sigma)]}{n} \sin(ny - n\pi D_0 \sigma) \quad \text{for} \quad 0 < \sigma < 1, \tag{5}
\]

where \(B_n = \exp[-\sigma(1/G_0 + n^{-1/2}D_0)]\). For the region \(\sigma > 1\) we can use the approximate solution:

\[
V = \sum_{n=1}^{\infty} \frac{1}{n} \frac{2B_n \sin(ny - n^{1/2}D_0 \sigma)}{(B_n + 1) \exp(\sigma B_n) - 1}, \tag{6}
\]

where \(B_n = n/G_0 + n^{1/4}D_0\).

We can see in the Fig. 1 and Fig. 2 that there is very good agreement between the approximate and numerical solutions. However, the use the Eq. (1) is limited by the plane wave condition (the cut-off frequency) for harmonics of distorted waves. Unfortunately, the higher harmonics do not satisfy this condition, so some radial distortions may appear and influence the level of some harmonics in the given point. Therefore it is better to use the KZK equation in this case. The results were verified experimentally.

FIGURE 1: Comparison of the approximate and numerical solution of Eq. (1) for the first 5 harmonics and 10th harmonic. Solid curves represent the approximate solution and dotted curves numerical solution, \(0 < \sigma < 1\).

FIGURE 2: Comparison of the approximate and numerical solution of Eq. (1) for the first 5 harmonics and 10th harmonic. Solid curves represent the approximate solution and dotted curves numerical solution, \(\sigma > 1\).

REFERENCES