Formulation of Boundary Value Problem
by a New Principle of Diffraction

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Abstract: A new diffraction principle that is formulated from a viewpoint centered at an observation point by considering
the wave propagation in a space seen by the observer virtually, has been proposed. This new approach allows us to grasp a
simple understandable concept of diffraction and to formulate the first rigorous and unique representation of diffracted waves,
that is, it satisfies both the wave equation and hard or soft boundary conditions for arbitrary 3-D obstacles. Thus the diffraction
problem for the scalar wave equation is solved by this approach. The representation is expressed in a form of integral
relation. Thus it is another problem to formulate the algorithm to find the solution. In this paper the boundary value problem is
formulated using the new principle.

INTRODUCTION

The representation of sound field by a new principle of diffraction called Virtual Discontinuity Principle of Diffraction
(abbreviated by VDPD) is expressed in a form of integral relation (1). Thus the variable in the integrand that relates to the sound field
is replaced by the geometrical-optics solution of sound field and by performing the integration the 1st order approximate solution is
obtained if the geometrical-optics solution is referred to the 0th order approximate solution. The 1st order solutions are compared with the experimental results using simple objects such as slit, cubes and concave objects composed by two square plates. The 1st order solutions describe
well the tendency of the experimentally obtained diffraction pattern but there remain some error between the estimation and experiment. Then we tried to improve the accuracy of estimation by replacing the geometrical-optics solution by the 1st order solution in the integrand to obtain the 2nd order solution and so on. The accuracy is improved by this process but it takes enormous computation time. Thus we could carry out this process only for the slit up to the 3rd order solution. It becomes clear that it is not practical to apply this process to 3D objects. In this paper the numerical procedures for the another approach, that is, the formulation of boundary value problem by the new principle of diffraction, will be described in details.

FORMULATION OF BOUNDARY VALUE PROBLEM

In order to formulate the boundary value problem by the new principle, let place the observation point O on the boundary of object P as shown in Fig. 1. Then a scalar potential \( U \) can be written as

\[
U(r_o) = U^g(r_o) + 2 \int_{D(O)} -g(r_o, r_D) \partial U(r_D)/\partial n_D \, dr_D
\]

where \( r_o \) is the position vector of O, \( r_D \) the position vector of a point on D(O), \( U^g \) the geometrical-optics solution, S a point source, D(O) the extension of the edge, on which O is placed, in the both sides, nD an outer normal to D(O), g the Green's function and is given by

\[
f(r_o, r_P) = \delta g(r_o, r_P)/\delta n_P
\]
Where $H_0^{(2)}$ is the 0th order Hankel function of 2nd kind and $k$ is the wavenumber. In order to formulate the boundary value problem, we need one more relation besides Eq.(1) since the function on $r_D$ must be related to a function on $r_P$. Let us apply the Green's theorem to the boundary of $P$. Then the following relation is obtained,

$$U(r_D) = U'(r_D) + \int_{r_P} U(r_P) \frac{\partial g(r_D, r_P)}{\partial n_P} \, dr_P$$  \hspace{1cm} (3)

where $U'$ is the direct incident wave on $r_D$, the position vector of a point on $P$, and $n_P$ is an outer normal to $P$. Eq.(3) has been used as a fundamental relation in BEM and will be referred later. If Eq.(3) is derived in the direction of $n_D$, then the following relation is obtained,

$$\frac{\partial U(r_D)}{\partial n_D} = \frac{\partial U'(r_D)}{\partial n_D} + \int_{r_P} U(r_P) \frac{\partial^2 g(r_D, r_P)}{\partial n_P \partial n_D} \, dr_P$$  \hspace{1cm} (4)

and if Eq.(4) is substituted into Eq.(1), then

$$U'(r_D) = \int_{r_D} \frac{\partial^2 g(r_D, r_P)}{\partial n_P \partial n_D} \, dr_D$$  \hspace{1cm} (5)

are obtained where the first two terms are constants that depend on the position of $O$ and $S$ and the third term expresses the contribution from $U$ on $P$. Eq.(6) shows the contribution of $U$ on $r_D$ to $U$ at $r_P$. Thus let call $f(r_D, r_P)$ as a response function of the potential. Eqs.(5) and (6) correspond to the formulation of boundary value problem by VDPD. In this formulation the key point lies on how to evaluate $f(r_D, r_P)$ in Eq.(6) since it is expressed as an integral along the half line $D(O)$ and $f(r_D, r_P)$ diverges as $r_D$ approaches to the edge points of $D(O)$. Thus the response function by VDPD is localized near the edges of $P$. It may be considered natural since the diffraction occurs at the edges. The localization of the response function will simplify the algorithm to solve the boundary value problem. On the other hand the response function in BEM is given by

$$g(r_D, r_P) = \frac{1}{4\pi} \frac{H_0^{(2)}(k|r_D-r_P|)}{\sqrt{r_D-r_P}}$$  \hspace{1cm} (6)

and in this case the response function is not localized at all.

Let us express Eq.(6) as follows

$$f(r_D, r_P) = \frac{\partial}{\partial n_P} \left( \int_{D(O)} g(r_D, r_P) \frac{\partial g(r_D, r_P)}{\partial n_D} \, dr_D \right)$$  \hspace{1cm} (7)

$$= \left( h(r_D, r_P) - h(r_D, r_P) \right) / e , \quad (e \to 0)$$  \hspace{1cm} (8)

$$h(r_D, r_P) = \int_{D(O)} g(r_D, r_P) \frac{\partial g(r_D, r_P)}{\partial n_D} \, dr_D$$  \hspace{1cm} (9)

where $e$ is a small increment in the $n_P$ direction. As can be seen in Eq.(10), $h(r_D, r_P)$ corresponds to the 1st order solution at $r_D$ where a point source of amplitude 1/4j is placed at $r_P$. Since the fast computational method for evaluating the 1st order solution precisely is available (2), it becomes possible to evaluate the response function precisely by Eq.(9).

The structure of the response function, that shows mutual dependence of the potential on the object, is made clear by VDPD and it reveals a special role of the edge point in the diffraction process. The usefulness of the response function will be shown at the congress by solving the boundary value problem for the potential on a semi-infinite plane.

REFERENCES