Lumped impedance of a planar discontinuity in an acoustic waveguide

Ralph T. Muehleisen\(^1\) and Anthony A. Atchley\(^2\)

\(^1\)Department of Civil, Environmental, and Architectural Engineering, University of Colorado, Boulder, CO 80309
\(^2\)Graduate Program in Acoustics, The Pennsylvania State University, University Park, PA 16802

Abstract: When a plane wave is incident upon an area discontinuity a pressure drop and phase shift results from viscous losses and the excitation of evanescent higher order modes. Using matrix methods, a closed form solution for the equivalent complex lumped acoustic impedance of the boundary can be derived. Using the resultant impedance model in the computation of the resonance frequency of a constricted annulus, the new impedance is found to be more accurate than one derived from conformal mapping.

INTRODUCTION

When a plane wave is incident upon a discontinuity, part of the wave is transmitted through the discontinuity and part is reflected from it. The standard boundary conditions that usually used to solve the problem are continuity of plane wave pressure and volume velocity\(^1\). However, below the cutoff frequency of the waveguide, evanescent higher order modes are excited at the junction which modify the magnitude and phase of the transmitted and reflected plane waves. The effects of the evanescent modes can be modeled through the use of a reactive impedance element. While thermal losses can usually be neglected, viscous losses at the discontinuity can be incorporated through the use of a resistive impedance element. Combining the two, the effects of a planar discontinuity area on plane wave propagation can be modeled using complex lumped impedance.

MODEL DERIVATION

Consider an acoustic waveguide with a planar area discontinuity as shown in Figure 1. A plane wave with pressure amplitude \(A_{10}\) is incident upon the junction in a waveguide with area \(S_1\). Plane wave and higher order modes \(B_{1M}\) are reflected. Planes waves and higher order modes \(A_{2M}\) are transmitted through the junction into a waveguide with area \(S_2\). The pressure in the waveguides can be written as

\[
p_1 = (A_{10} e^{-j\beta_1 x} + B_{10} e^{j\beta_1 x}) \Phi_{10} + \sum_{M=1}^{\infty} B_{1M} e^{j\gamma_{1M} x} \Phi_{1M},
\]

\[
p_2 = A_{20} e^{-j\beta_2 x} \Phi_{20} + \sum_{M=1}^{\infty} A_{2M} e^{j\gamma_{2M} x} \Phi_{2M},
\]

where \(\Phi_{1M}\) is the mode shape, \(k_1\) is the plane wavenumber (including damping), \(\gamma_{1M} = \sqrt{\chi_{1M}^2 - k_1^2}\) is the propagation constant, \(\chi_{1M}\) is the eigenvalue of the mode shape \(\Phi_{1M}\), and \(\int \Phi_{1M} \Phi_{1N} dS_1 = \delta_{MN}\).

The boundary conditions at \(x=0\) are that volume velocity is conserved and that the plane wave pressure differs by the pressure drops due to higher order mode excitation and viscous losses. A complex impedance multiplied by the volume velocity through the junction can model these pressure drops. If the equations for the boundary conditions are expressed in matrix form, a closed form solution can be obtained. Let us begin by defining the mutual modal coupling coefficient as

\[
H_{MN} = \int_{S_2} \Phi_{1M}(y, z) \Phi_{2N}(y, z) dS_2.
\]

If \(S_w\) is the rigid area outside \(S_2\) at \(x=0\) (\(S_w=S_1-S_2\)), the tangential derivative coefficients are
Defining the modal impedances as $Y_{im} = j\lambda_{im} / \rho_0 c k_i$ and $Y_{i0} = 1 / (\rho_0 c)$, the higher order mode matrices are

$$A = \begin{bmatrix} A_{1,1} & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ \end{bmatrix}, \quad B = \begin{bmatrix} B_{1,1} & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ \end{bmatrix}, \quad A_2 = \begin{bmatrix} A_{2,1} & \cdots & \cdots & \cdots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ \end{bmatrix}, \quad H_{M0} = \begin{bmatrix} H_{1,0} \\ \vdots \\ \vdots \\ \vdots \\ \end{bmatrix}, \quad H = \begin{bmatrix} H_{1,1} & H_{1,2} & \cdots \\ \vdots & \ddots & \cdots \\ \vdots & \ddots & \cdots \\ \vdots & \ddots & \cdots \\ \end{bmatrix}.$$  

$$Y_1 = \begin{bmatrix} Y_{1,1} \\ \vdots \\ \vdots \\ \vdots \\ \end{bmatrix}, \quad Y_2 = \begin{bmatrix} Y_{2,1} \\ \vdots \\ \vdots \\ \vdots \\ \end{bmatrix}, \quad J_y = \begin{bmatrix} J_{y1,1} & J_{y1,2} & \cdots \\ \vdots & \ddots & \cdots \\ \vdots & \ddots & \cdots \\ \vdots & \ddots & \cdots \\ \end{bmatrix}, \quad J_z = \begin{bmatrix} J_{z1,1} & J_{z1,2} & \cdots \\ \vdots & \ddots & \cdots \\ \vdots & \ddots & \cdots \\ \vdots & \ddots & \cdots \\ \end{bmatrix}.$$  

Applying the pressure and volume velocity boundary conditions, equating the viscous power loss to the power dissipated by the lumped impedance (neglecting thermal losses), and defining $Z_{eq} = [Y_{1} + H Y_{2} H^T]^{-1}$ as the complex conjugate operator, and $\delta_u$ as the viscous penetration depth, the expression for the lumped element impedance of the junction is

$$Z_{\text{junction}} = \frac{\delta_u}{2\rho_0 c k_i S_2} \left( H_{M0}^T Z_{eq} (J_y + J_z) Z_{eq} H_{M0} + H_{M0}^T Z_{eq} H_{M0} \right).$$  

APPLICATION

Consider as an application, calculation of the resonance frequencies of a constricted annular resonator. Table 1 shows a comparison of the error between the measured resonance frequency to the resonance frequency calculated without correction, applying a flanged end correction to the ends of the constriction, using a lumped impedance derived by conformal mapping, and by using the formula above. The frequency of resonance was well below the cutoff frequency of the first higher order mode. While the conformal mapping derived impedance works well for larger area ratios, overall it is not as accurate as the impedance derived above using higher order modes.

<table>
<thead>
<tr>
<th>Area Ratio</th>
<th>No Correction</th>
<th>Flanged Correction</th>
<th>Conformal Mapping</th>
<th>Higher Order Mode</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>10.7%</td>
<td>0.8%</td>
<td>1.8%</td>
<td>0.3%</td>
</tr>
<tr>
<td>0.2</td>
<td>7.1%</td>
<td>2.9%</td>
<td>0.3%</td>
<td>0.1%</td>
</tr>
<tr>
<td>0.3</td>
<td>4.7%</td>
<td>4.0%</td>
<td>0.1%</td>
<td>0.1%</td>
</tr>
</tbody>
</table>

CONCLUSIONS

When a plane wave is incident on a planar junction, there is a plane wave pressure drop and phase shift from the viscous losses at the wall and excitation of higher order modes. By writing the resulting modal equations in matrix form a closed form solution for the lumped impedance can be computed. Computation of the resonance frequency of a constricted annular resonator shows the accuracy of the model. The model can be calculated for the junction of waveguides of any shape as long as the mode shapes are known.

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REFERENCES