Chaotic dynamics in acoustics

Werner Lauterborn

Drittes Physikalisches Institut, Universität Göttingen, Bürgerstraße 42-44, D-37073 Göttingen, Germany

Abstract: The theory developed for the description of chaotic dynamical systems also pertains to acoustics. To apply the new ideas and methods experimentalists have to resort to embedding their data (time series) into high dimensional spaces. Some questions pertaining to the embedding procedure are addressed and examples of chaotic acoustical systems are given.

A number of genuine phenomena occurring in nature calls for nonlinearity to be described properly. Among them, chaotic dynamics is the most puzzling and intriguing one. It has been found that systems with chaotic dynamics abound in nature, and examples can be drawn from almost any scientific discipline, notably also acoustics (1). Acoustic cavitation, for instance, has supplied one of the first experimental examples where period-doubling to chaotic motion has been found and methods of nonlinear dynamics have been exemplified (2,3). Further acoustical systems that have been investigated with these methods are vibrated liquid surfaces (Faraday experiment), musical instruments, nonlinear oscillators – in particular the thermoacoustic oscillator and the bubble oscillator --, speech and hearing, and acoustic cavitation (4).

STATE SPACE, ATTRACTORS AND EMBEDDING

The state of a dynamical system is most intuitively visualized geometrically as a point in a space, called state space or phase space, spanned by the dependent variables of the system. Variables may be voltages, currents, temperatures, pressures, etc. The dynamic evolution, i.e., the change of the system with time, then is given by a curve in the state space of the system, called trajectory or orbit. In dissipative systems as they are usually encountered in physics the trajectories head for a limit set called attractor. There exist different types of attractors. The most simple one is a fixed point, i.e., an equilibrium or state of rest. The next, more involved attractor, is called a limit cycle. It is a closed curve in state space that is traversed again and again by the system. Self-excited oscillators, for instance, belong to this class. The next simple attractor fills a two-dimensional manifold, a torus. A trajectory on the torus is a quasiperiodic motion. It comes about through two incommensurable frequencies being present simultaneously. The most intriguing and variable type of attractor is the strange or chaotic attractor. It has a broad band spectrum like noise and is better characterized via its fractal dimension (static property) and its Lyapunov spectrum (dynamic property). To demonstrate its real existence Fig. 1, left, shows a stroboscopic phase portrait of a strange attractor, a chaotically oscillating driven pendulum from a real experiment.

FIGURE 1. Left: Chaotic attractor from the driven pendulum (Courtesy of M. Kaufmann). Right: Three reconstructions with different delay times \( t_i \) for experimental data obtained from a chaotically oscillating, periodically driven pendulum.

In mathematical models of dynamical systems the dynamics is visualized in their state space, whose (integer) dimension is given by the number of the dependent variables of the model. In experiments, usually just one variable is measured as a function of time, and the state space is not known. How, then, to arrive at the attractor that may characterize the system? It has been found that a surprisingly simple procedure yields a reconstructed state space in the following way (3,5). The scalar signal \( s(t) \) is
sampled with a sampling time $t_e$. The resulting time series $\{s^n\}$ with $s^n = s(nt_e)$ is used to construct the states $y^n = (s^n, s^{n+t}, s^{n+2t}, ..., s^{n+(d-1)t})$ for $n = 1, ..., N$. The symbol $l$ denotes the delay time or lag in units of the sampling time (i.e., $t_l = lt_e$), $d$ is the embedding dimension.

The reconstruction using delay coordinates is based on two parameters: the embedding dimension $d$ and the delay time $t_l$. When the dimension $d$ is chosen too small the conditions given in the embedding theorems are not fulfilled. For $d$ too large, on the other hand, practical problems occur due to the fixed amount of points that constitute thinner and thinner sets in the $d$-dimensional space. When the delay time $t_l$ is too small, the coordinates $y^n_l = s^{n+(i-1)t_l}$ of each reconstructed state $y^n$ do not significantly differ from one another and therefore the state points are scattered along the diagonal (Fig. 2a). In this case, any investigation of a possible fractal structure of the attractor (e.g., a dimension estimation) becomes difficult. The situation improves when $t_l$ is increased. Then the attractor unfolds and its inner structure becomes "visible" more readily as can be seen in Fig. 2b. When the delay time $t_l$ is increased further, the reconstructed attractor is folded more and more (Fig. 2c). After the embedding has taken place, the fractal dimension of the point set obtained and the Lyapunov spectrum may be calculated (3,5). The (fractal) dimension of an attractor characterizes its complexity and gives a lower bound for the number of equations or variables needed for modeling the underlying dynamical process. The Lyapunov spectrum is the set of Lyapunov exponents of the system describing the expansion or contraction properties of the system. If the largest one is positive the system is said to be chaotic.

**CHAOTIC ACOUSTICAL SYSTEMS**

Acoustic systems that have been investigated this way are still rare. One system under active investigation is the vibrated liquid layer (Fig. 2, left), the Faraday experiment, where regular and chaotic patterns appear on the surface. From the musical instruments the woodwind-like instruments are presently under intense study (6). The most complete investigation into chaotic dynamics has been done in the area of acoustic cavitation (see Fig.2, right), where bubbles are produced and set into nonlinear oscillations in a liquid by ultrasonic waves (2,4,7). In general, chaotic dynamics is to be expected everywhere, where a nonlinear, dissipative system is driven. And that happens often in acoustics.

**FIGURE 2.** Left: Vibrated liquid layer. Right: Piezoelectric cylinder for acoustic cavitation.

**REFERENCES**